Predictive control and synchronization of the four-dimensional energy resource chaotic system

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Abstract—This paper addresses the control and synchronization problems of the four-dimensional energy resource chaotic system. The proposed method combines the delayed feedback control of high order continuous-time chaotic systems with the prediction-based method of discrete-time chaotic systems. Then the controllers for control and synchronization are designed using some new and useful criteria derived from the matrix theory concept. Finally, numerical simulations are presented to illustrate the effectiveness and feasibility of the theoretical results.

Keywords: Four-dimensional energy resource chaotic system, predictive control, predictive synchronization, matrix measure.

I. INTRODUCTION

Chaos as a very interesting nonlinear phenomenon has been widely investigated in science and engineering. Recent researches have demonstrated that chaos can be useful in practical applications such as lasers, biology, economics, chemical reactions, and secure communications. However, to take full advantage of chaos, it is necessary to control it. Several strategies to control chaos have been proposed and investigated with the objective of stabilizing equilibrium points or periodic orbits embedded in chaotic attractors. It is noticed that the stabilization of unstable equilibrium points is mostly used in engineering applications, which has many advantages such as guaranteed and robust stability, precise target tracking, and strong ability of noise rejection. For example, stable equilibria can be useful for making predictions because lots of solutions eventually settle down near the stable equilibria.

On the other hand, controlling chaos has emerged as a new and a very attractive subject to study, and has developed many theories and methodologies since the pioneering work of Ott, Grebogi and York [1]. We report some well-known chaos control methods such as time delay feedback method [2], adaptive control method [3], occasional proportional feedback method [4], impulsive control method [5], higher-order method [6], predictive control method [7], [8], generalized control by nonlinear high-order approach [9], and so on [10]–[16]. Among them, predictive control is especially attractive due to its simplicity in configuration and implementation. In view of its advantages, predictive control has been successfully applied to the stabilization of various chaotic systems [17]–[19], and to the synchronization of chaotic satellite systems [20]. From all obtained results, the stabilization and synchronization of the chaotic systems are achieved with very satisfactory performances.

Motivated by the above discussion, we are interested in this paper with the control and synchronization of the four-dimensional energy resource chaotic system by using predictive control method. In order to demonstrate the effectiveness of the proposed approach, one of the main contributions of this paper is to present numerical results on high dimensional chaotic systems.

The rest of the paper is organized as follows. In Section 2, the concept of the predictive control is presented. Based on this concept, control and synchronization problems are addressed. The stability and synchronization of four-dimensional energy resource chaotic system are described in Section 3. Numerical simulations are included to demonstrate theoretical results. Finally, concluding remarks are given in Section 4.

II. PREDICTIVE CONTROL AND SYNCHRONIZATION PRINCIPLES

In this section, we apply the predictive control algorithm for the four-dimensional energy resource chaotic system, in which the gain matrix of the state feedback controller is determined from the new sufficient condition for global asymptotic stability. The sufficient condition is derived using the matrix measure theory. Comparing with previous results, the proposed method allows an improvement in system performance in the sense of reducing the feedback gain matrix to a diagonal form. The problem of the upper bound is formulated using the induced matrix norm.

A. Predictive control principle

Predictive control methods may be considered as a kind of adaptive control strategy. Among different methods, Ushio and Yamamoto [7] proposed a chaos control method of discrete-time systems based on the prediction of τ-time future state and the control law is calculated from the difference between the current state and the τ-time future state of the chaotic system. It was shown that this method guarantees the stability of the obtained controlled system. Considering a chaotic system with state equation under control input $u$ in the form

$$\dot{X}(t) = f(X(t)) + u(t) \quad (1)$$

where $X \in R^N$ is the state vector, $f : R^N \rightarrow R^N$ is a continuous vector function, and $u \in R^N$ is the control action. We assume that $f(\cdot)$ is differentiable and the system is
generating chaos when \( u(t) = 0 \). The task is to find a control \( u(t) \) such that the controlled system trajectory tracks the target
\[
\lim_{t \to \infty} \| X(t) - E \| = 0
\]  
(2)
where \( E \) is the unstable equilibrium point to be stabilized.

The proposed predictive control method is based on the prediction of a \( \tau \)-time future state and the control law is calculated from the difference between the current state and the future state of the uncontrolled four-dimensional energy resource chaotic system. We choose the feedback control input as follows
\[
u(t) = K(AX(t) - X(t))
\]  
(3)
where \( A \) represents the system’s Jacobian matrix and \( K = \text{diag}(k_1, k_2, \ldots, k_n) \) is a feedback matrix to be designed later.

Eq. (3) is exactly the control proposed by Boukabou et al. [8] to avoid some handicaps of the delay feedback control [2] and the predictive feedback control of discrete-time chaotic systems [7], such as the increase in dimensionality and the so-called odd limitation number. Some additional good properties of the control law (3) underlined in [8] are that the equilibrium points of the controlled system are the same as those in the uncontrolled system, and it is easy to implement in the sense that the control term contains only the amplified versions of the input and output of the dynamical system. In our main result, we prove that for the four-dimensional energy resource chaotic system, the proposed modified predictive control using matrix measure leads to the global stabilization of the system into its unstable equilibrium points.

Now, define \( y(t) = X(t) - E \), then the closed-loop system from (1) and (3) can be written as
\[
\dot{y}(t) = Ay(t) + K(Ay(t) - y(t))
\]  
(4)
thus we obtain
\[
\dot{y}(t) = (A + K(A - I))y(t).
\]  
(5)
where \( I \) is the identity matrix.

Note that the origin is the equilibrium point of system (5), and our aim is to stabilize system (5) by designing the control gain matrix \( K \) to achieve the goal (2).

In the following, we introduce the concept of matrix measure which is useful for the next theorem.

**Definition 1:** The matrix measure of a real square matrix \( A = (a_{ij})_{N \times N} \) is as follows [21]:
\[
\mu_1(A) = \lim_{\varepsilon \to 0^+} \frac{\| I + \varepsilon A \|_i - 1}{\varepsilon},
\]  
(6)
where \( \| . \|_i \) is an induced matrix norm on \( \mathbb{R}^{N \times N} \), \( I \) is the identity matrix, and \( i = 1, 2, \infty \).

The measure of a matrix \( \mu_1(A) \) is considered as the directional derivative of the induced matrix norm \( \| . \|_i \), as evaluated at the identity matrix \( I \) in the direction \( A \).

**Example 2:** For the matrix norms:
\[
\| A \|_1 = \max \sum_{i=1}^{n} |a_{ij}|, \| A \|_2 = \sqrt{\lambda_{\max}(A^TA)},
\]  
and
\[
\| A \|_{\infty} = \max \sum_{j=1}^{n} |a_{ij}|,
\]  

the corresponding matrix measures are given by
\[
\mu_1(A) = \max \left( a_{jj} + \sum_{i=1, i \neq j}^{n} |a_{ij}| \right), \mu_2(A) = \frac{1}{2} \lambda_{\max}(A^TA),\text{ and } \mu_\infty(A) = \max \left( a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ij}| \right),
\]  
respectively.

**Theorem 3:** If there exists a feedback gain matrix \( K \) such that
\[
\mu_i(A + K(A - I)) < 0, \quad i = 1, 2, \infty,
\]  
(7)
then the zero solution of system (5) is globally asymptotically stable, implying that the four-dimensional energy resource chaotic system converges towards the unstable equilibrium point \( E \).

**Proof.** Let us define a Lyapunov function of the form \( V(y(t)) = \| y(t) \|_i \), then the upper right-hand derivative of \( V(y(t)) \) with respect to time is as follows:
\[
D^+ V(y(t)) = \lim_{h \to 0^+} \frac{\| y(t + h) \|_i - \| y(t) \|_i}{h}.
\]

Based on Definition 1, we have
\[
D^+ V(y(t)) \leq \mu_i(A + K(A - I)) \| y(t) \|_i = -\alpha V(y(t))
\]
where \( \alpha = -\mu_i(A + K(A - I)) \). From Eq. (7), \( \alpha > 0 \), thus \( V(y(t)) \leq V(y(0))e^{-\alpha t} \). Therefore, we deduce that the condition of Theorem 1 ensures global asymptotic stability of system (5), i.e., \( X(t) \) converges to \( E \) as \( t \to \infty \). This completes the proof.

**Remark 4:** All the equilibrium points of the four-dimensional energy resource chaotic system are stabilized by the predictive control method if there exists a gain matrix \( K \) such that (7) is verified and if and only if \( \det(A - I) \neq 0 \).

**Corollary 5:** The controlled four-dimensional energy resource chaotic system (5) is globally asymptotically stabilized if there exists the feedback gain matrix \( K \) such that one condition is satisfied at least as follows:
\[
\max_i \left( a_{ii} + k_i(a_{ii} - 1) + \sum_{j=1, j \neq i}^{n} |a_{ij}| \right) < 0, \quad \max_j \left( a_{jj} + k_j(a_{jj} - 1) + \sum_{i=1, i \neq j}^{n} |a_{ij}| \right) < 0, \quad \lambda_{\max} \left( (A + K(A - I))^T + (A + K(A - I)) \right) < 0
\]  
(8)
where \( \lambda_{\max} \) is the largest eigenvalue of the matrix between brackets.

**Proof.** It is easy to prove the two first inequalities according to Theorem 1 and Example 1. For the last inequality, let us
define a Lyapunov function of the form
\[ V(x) = X^T(t)PX(t) \]
where \( P \) is a symmetric constant matrix. Then the time derivative of \( V \) is given by
\[
\dot{V}(X(t)) = X^T(t)PX(t) + X^T(t)P\dot{X}(t) \\
= X^T(t)[A + K(A - I)]^TPX(t) + X^T(t)P[A + K(A - I)]X(t) \\
= X^T(t)[A^TP + PA + (A - I)K^TP + PK(A - I)]X(t) \\
\leq \mu_i[\lambda_{\text{min}}(P)] \{\mu_i[A^TP + PA + (A - I)K^TP + PK(A - I)]V(X(t)) \}
\]
where \( \lambda_{\text{min}}(P) \) is the smallest eigenvalue of matrix \( P \). This completes the proof.

Remark 6: There are many papers concerning control and synchronization in complex networks. In Refs. [22] and [23], some control and synchronization criteria are presented based on the concept of pinning control. The proposed predictive control using matrix measure in this paper can also be applied to study control and synchronization in complex networks. In fact, matrix measure can have positive as well as negative values (\( \mu_i(-A) \neq \mu_i(A) \)), whereas a norm can assume only non-negative values (\( \|A\|_1 = \|A\|_\infty \)).

B. Predictive synchronization principle

Let \( X_1(t), X_1(0) \) and \( X_2(t), X_2(0) \) be solutions to the master system and to the slave system, respectively.

In this framework, complete synchronization is defined as the identity between the trajectories of the slave system \( X_2(0) \) and of one replica \( \tilde{X}_2(0) \) of it \( \tilde{X}_2(t) = g(X_1(t), X_2(t)) \) for the same chaotic master system.

Let us define the error dynamical system as follows
\[ \dot{e}(t) = \dot{X}_1(t) - \dot{X}_2(t), \]
(9)
Then, the master system and the slave system are said synchronized if and only if:
\[ \lim_{t \to \infty} \|e(t)\| = 0. \]
(10)
In other words, the slave system forgets its initial conditions, though evolving on a chaotic attractor. Hence, the synchronization objective is to force \( X_2(t) \to X_1(t) \) as \( t \to \infty \).

The closed-loop error system can be written as
\[ \dot{e}(t) = Ae(t) + K(Ae(t) - e(t)) \]
(11)
thus we obtain
\[ \dot{e}(t) = (A + K(A - I))e(t), \]
(12)
where \( I \) is the identity matrix. Using the matrix measure theory given by Theorem 1, the closed-loop error system is globally

### III. Simulation results

In this section, we will give results of simulation for the predictive control of the four-dimensional energy resource system.

#### A. System description

The four-dimensional energy resource chaotic system is given by the following set of ordinary differential equations [24]:
\[
\begin{align*}
\dot{x} &= a_1x(1 - x/M) - a_2(y + z) - d_3w \\
\dot{y} &= -b_1y - b_2z + b_3x[N - (x - z)] \\
\dot{z} &= c_1z(c_2x - c_3) \\
\dot{w} &= d_1x - d_2w
\end{align*}
\]
(13)
where where \( x(t), y(t), z(t), \) and \( w(t) \) are the energy resource shortage in A region, the energy resource supply increment in B region, the energy resource import in A region, the renewable energy resources in A region, respectively. Parameters \( M, N, a_i, b_j, c_j, d_j \) \((i = 1, 2, j = 1, 2, 3)\) are all positive real. When \( a_1 = 0.09, a_2 = 0.15, b_1 = 0.06, b_2 = 0.083, b_3 = 0.07, c_1 = 0.2, c_2 = 0.5, c_3 = 0.4, d_1 = 0.1, d_2 = 0.06, d_3 = 0.08, M = 1.8, N = 1.0, \) this system exhibits a chaotic behavior as shown in Fig. 1.

In order to presents the evolution of the chaotic system in function of one the system parameters, we have carried out simulation by varying the parameter \( d_1 \) from 0 to 10 with an iteration step of 0.01. The result of this procedure is that the resulting bifurcation diagram is reminiscent of the logistic and other quadratic functions.

Also, we can see that from a main branch, we get two branches then these last branches are subdivided into two other branches. Every division corresponds to a period doubling of the function. We have a stable fixed point for \( 0.15 < d_1 < 0.2 \), the 2-cycle periodic orbit for \( 0.15 < d_1 < 0.15 \), the stable 4-cycle periodic orbit appears at approximately \( 0.15 < d_1 < 0.145 \), also through \( d_1 < 0.135 \) where appears chaos.

#### B. Control result

To stabilize the four-dimensional energy resource chaotic system on the desired unstable equilibrium point, a predictive controller in the form of (3) is designed. For the choice \( K = \text{diag}(-1, 0, 0, 0) \), the four-dimensional energy resource chaotic system states converge towards the unstable equilibrium points. Results of simulations are illustrated in Fig. 3. It is observed that the state vectors of the controlled system are asymptotically stabilized on the desired equilibrium point.

#### C. Synchronization result

In this case, we choose the initial values of the master system as \( x_1(0) = 0.82, y_1(0) = 0.29, z_1(0) = 0.48 \) and \( z_1(0) = 0.1 \) and the initial values of the slave system as \( x_2(0) = 1.6, y_2(0) = 0.59, z_2(0) = 0.74 \) and \( w_2(0) = 0.34 \).
IV. CONCLUSION

In this paper, a predictive framework for controlling and synchronizing the four-dimensional energy resource chaotic system has been proposed. The proposed method has many advantages such as small impulse control and unified control way. Some new and less conservative criteria have been derived based on matrix measure to guarantee the global asymptotic stability of the system. Numerical simulations are given to show the effectiveness of the proposed method.
REFERENCES